Monotone Subsequences in $\mathbb{R}^d$

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Abstract

This paper investigates the length of the longest monotone subsequence of a set of n points in $\mathbb{R}^d$. A sequence of points in $\mathbb{R}^d$ is called monotone in $\mathbb{R}^d$ if it is monotone with respect to some order from $\mathcal{R}_d = \{\leq, \geq\}^d$, with other words if it is monotone in each dimension $i \in \{1, \ldots, d\}$. The main result of this paper is the construction of a set P which has no monotone subsequences of length larger then $\lceil n^{\frac{1}{2^d-1}} \rceil$. This generalizes to higher dimensions the Erdös-Szekeres result that there is a 2-dimensional set of n points which has monotone subsequences of length at most $\lceil \sqrt{n} \rceil$.

1 Introduction

Erdös and Szekeres [4] proved that any sequence $\{a_j\}$ of n real numbers has a monotone (increasing or decreasing) subsequence of length $\lceil \sqrt{n} \rceil$. They pointed out that there exists a sequence of n distinct real numbers which has monotone subsequences of length at most $\lceil \sqrt{n} \rceil$. This is equivalent to the fact that there is a 2-dimensional set of n points which has monotone subsequences of length at most $\lceil \sqrt{n} \rceil$, because any one-dimensional sequence $\{a_j\}$ can be seen as a 2-dimensional set $\{(i, a_i) \mid i \in \{1, \ldots, n\}\}$ of n distinct points in $\mathbb{R}^2$. Now considering a set $S = \{(a_i, b_i) \mid i \in \{1, \ldots, n\}\}$ of n distinct points we can easily find a monotone subsequence of length $\lceil \sqrt{n} \rceil$: we can sort the elements of S with respect to the increasing order of the first coordinate of the points; w.l.o.g. let this order be $a_1 \leq a_2 \leq \ldots \leq a_n$. The sequence $\{b_i\}$ has a monotone subsequence $\{b_{i_j}\}$ of length $\lceil \sqrt{n} \rceil$. Thus the subsequence $\{(a_{i_j}, b_{i_j})\}$ of S of length $\lceil \sqrt{n} \rceil$ is monotone with respect to some order $o \in \{ (\leq, \leq); (\leq, \geq) \}$.

This paper generalizes the Erdös-Szekeres result to higher dimensions. It investigates the length of the longest monotone subsequence of a set of n points in $\mathbb{R}^d$. A sequence of points in $\mathbb{R}^d$ is called monotone in $\mathbb{R}^d$ if it is monotone with respect to some order from $\mathcal{R}_d = \{\leq, \geq\}^d$, with other words if it is monotone in each dimension $i \in \{1, \ldots, d\}$. The main result of this paper is the construction of a set P of n points which has no monotone subsequences of length larger then $\lceil n^{\frac{1}{2^d-1}} \rceil$. Note that any set of n points in $\mathbb{R}^d$ has a monotone subsequence of length at least $\lceil n^{\frac{1}{2^d-1}} \rceil$ (see Section 3).

Siders investigates in [10] another possibility to generalize the Erdös-Szekeres result to higher dimensions : he constructs a sequence of n points in d dimensions such that, when

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projected in a general direction, the sequence has no (one-dimensional) monotone subsequences of length $\sqrt{n}+d$ or more.

A consequence of the Erdös-Szekeres result is the existence of a partition of a set of n points in the plane into $O(\sqrt{n}\,)$ monotone subsequences. The best known algorithm for computing such a partition runs in time $O(n^{3/2})$ [1]. A longest monotone increasing subsequence of a sequence of n real numbers can be computed in time $O(n\log n)$. Felsner and Wernisch give in [5] an algorithm that computes maximum k increasing subsequences in time $O(kn\log n)$.

Partitioning into monotone subsequences is a useful tool for various applications in the plane. Matousek and Welzl give in [8] an algorithm for the halfspace range-counting problem in the plane, using the Erdös-Szekeres result. This technique has also been applied in [2] to solve some other geometric-searching problems, including ray shooting and intersection searching.

The result of this paper shows that there are sets of points in $\mathbb{R}^d$ with very short monotone subsequences, thus partitioning into monotone subsequences in $\mathbb{R}^d$ may not be a promising tool for solving high dimensional geometric-searching problems. An interesting problem is what is the expected size of the longest monotone subsequence of a set of d-dimensional n points chosen at random from the unit cube $[0,1,]^d$ under uniform distribution. In the case of one-dimensional sequences Hammersley showed in [7] that the expected length of a maximum increasing subsequence in a random permutation of $\{1,2,\ldots,n\}$ converges to $c\sqrt{n}$ with increasing n, for some constant c. A simple proof that $c\leq 2$ is given by Pilpel in [9]. A review on the length of the longest increasing subsequence of n real numbers, which covers results on random and pseudo-random sequences is given in [11]. For the case of higher dimensions, Bollobás and Winkler proved in [3] that for n points, independent und uniformly distributed on $[0,1]^d$, the length of a longest subsequence which is monotone with respect to the *dominance order* [1] converges to $c_d\cdot\sqrt[d]{n}$ with increasing n, where c_d is a constant depending on d with $\lim_{d\to\infty} c_d = e$.

This paper is structured as follows. In Section 2 we introduce some basic notations and definitions. Section 3 presents a simple algorithm which finds a monotone subsequence of length at least $\lceil n^{\frac{1}{2^{d-1}}}\rceil$ in a set of n points in $\mathbb{R}^d$. Finally, in Section 4 we construct a set P of n points in $\mathbb{R}^d$ with longest monotone subsequences of length $\lceil n^{\frac{1}{2^{d-1}}}\rceil$.

2 Preliminaries

We define the set $\mathcal{R}_d = \{\leq,\geq\}^d$ of *reflexive partial orders* on $\mathbb{R}^d$. Let $o\in\mathcal{R}_d$, $o=(o(1),\ldots,o(d))$ with $o(i)\in\{\leq,\geq\}$, $i\in\{1,\ldots,d\}$. Consider two points in $\mathbb{R}^d$:

$$\begin{aligned} a &= (a_1,\ldots,a_d) \\ b &= (b_1,\ldots,b_d) \end{aligned}$$

where $a_i, b_i \in \mathbb{R}$. We write as usual $a\,o\,b$ to mean that (a,b) is in the order o, which is defined as follows:

$$a\ o\ b \quad \Longleftrightarrow \quad a_i\ o(i)\ b_i \ \ \forall\, i\in\{1,\ldots,d\}$$

where

$$a_i\, o(i)\ b_i \ = \ \begin{cases} a_i \leq b_i & \text{if } o(i) \ = \leq \\ a_i \geq b_i & \text{if } o(i) \ = \geq \end{cases}$$

[1] The dominance order is the partial order $<<$ on $\mathbb{R}^d$ such that $a=(a_1,\ldots,a_d)\ <<\ b=(b_1,\ldots,b_d)$ if and only if $a_i\leq b_i$ for each $i=1,\ldots,d$.

Analogously, we define the set $\mathcal{O}_d = \{<, >\}^d$ of *irreflexive partial orders* on $\mathbb{R}^d$.

Definition 2.1 A sequence $\mathcal{S} = [\ p_1, p_2, \ldots, p_r\ ]$ of distinct points from $\mathbb{R}^d$ is *monotone* in $\mathbb{R}^d$ if and only if there is some order $o \in \mathcal{R}_d \cup \mathcal{O}_d$ such that $p_1\ o\ p_2\ o\ \ldots\ o\ p_r$ holds. We call $\mathcal{S}$ to be *monotone with respect to o.*

Notation 2.1 Given $o = (o(1), \ldots, o(d)) \in \mathcal{R}_d \cup \mathcal{O}_d$ we denote by

$$\overline{o} = \left(\overline{o(1)}, \ldots, \overline{o(d)}\right)$$

where $\overline{\leq}\ = \geq$, $\overline{\geq}\ = \leq$, $\overline{<}\ = >$ and $\overline{>}\ = <$.

Definition 2.2 Given two different strings $x = [\ x_1x_2\ldots x_r\ ]$ and $y = [\ y_1y_2\ldots y_r\ ]$ where $x_j, y_j \in M \subset \mathbb{R}$, we say that string x is **lexicographically less than** string y if there exists an integer i, $0 \leq i \leq r$, such that $x_j = y_j$ for all $j = 0, \ldots i-1$ and $x_i\ <\ y_i$.

Throughout this paper we say that a set P has a monotone subsequence $\mathcal{S}$ with respect to some order $o \in \mathcal{R}_d$ if the set of all elements of $\mathcal{S}$ is a subset of P and the sequence $\mathcal{S}$ is monotone with respect to o.

3 Finding a monotone subsequence of length $\lceil n^{\frac{1}{2^{d-1}}}\ \rceil$

In this section we present a simple algorithm which finds a monotone subsequence of length at least $\lceil n^{\frac{1}{2^{d-1}}}\ \rceil$ in a set of n points in $\mathbb{R}^d$.

Notation 3.1 Let $\mathcal{S} = [\,p^1, p^2, \ldots, p^r\,]$ be a sequence of points, where $p^j = (p^j_1, p^j_2, \ldots, p^j_d) \in \mathbb{R}^d$. We denote by $\mathcal{S}(i)$ for $i \in \{1, \ldots, d\}$ the sequence $\mathcal{S}(i) = [\ p^1_i, p^2_i, \ldots, p^r_i\ ]$ of the coordinates in the i-th dimension of the points in $\mathcal{S}$.

There is a "folklore" algorithm for finding a monotone subsequence of a sequence of n reals in time $O(n \log n)$ (see e.g. [6], [8]). We will iteratively use this algorithm in order to find a monotone subsequence in a set P of n points in $\mathbb{R}^d$.

Let S_1 be the sequence of the n points ordered with respect to the increasing order of the first coordinate of the points. Now the sequence $S_1(2)$ of the n coordinates in the 2-nd dimension of S_1 has a monotone subsequence $S_2(2)$ of length $f_2(n) \geq \lceil \sqrt{n}\ \rceil$. Let S_2 be the d-dimensional sequence corresponding to $S_2(2)$. S_2 is a subsequence of S_1 which is monotone with respect to the first 2 coordinates. Now having a subsequence S_m ($m \geq 2$) of P of length $f_m(n)$ which is monotone with respect to the first m coordinates, we can find as described above a subsequence S_{m+1} of S_m which is monotone with respect to the first $m+1$ coordinates and has length

$$f_{m+1}(n) \geq \left\lceil \sqrt{f_m(n)}\ \right\rceil.$$

We repeat iteratively this procedure until we obtain the subsequence S_d of P which is monotone with respect to all d coordinates.

We have :

$$f_1(n) = n \ \text{ and } \ f_m(n) \geq \left\lceil \sqrt{f_{m-1}(n)}\ \right\rceil$$

We proof by induction on n that $f_m(n) \geq \lceil\ n^{\frac{1}{2^{m-1}}}\ \rceil$ holds using the following equation

$$\lceil \sqrt{\lceil x \rceil}\ \rceil = \lceil \sqrt{x}\ \rceil \quad \forall\, x \geq 0, x \in \mathbb{R} \tag{1}$$

The base of induction is trivial since $f_1(n) = \lceil n^{\frac{1}{2^0}}\ \rceil$ holds. For the induction step we assume that $f_m(n) \geq \lceil\ n^{\frac{1}{2^{m-1}}}\ \rceil$ holds for some $m \geq 1$. Let x equal $n^{\frac{1}{2^{m-1}}}$ in (1). Then we have:

$$f_{m+1}(n) \geq \left\lceil \sqrt{f_m(n)}\ \right\rceil \geq \left\lceil \sqrt{\lceil n^{\frac{1}{2^{m-1}}} \rceil}\ \right\rceil = \left\lceil\ \sqrt{n^{\frac{1}{2^{m-1}}}}\ \right\rceil = \left\lceil\ n^{\frac{1}{2^m}}\ \right\rceil$$

This implies that the length $f_d(n)$ of the subsequence S_d found by the procedure described is at least $\lceil n^{\frac{1}{2^{d-1}}} \rceil$.

4 Construction of a set with longest monotone subsequences of length $\lceil n^{\frac{1}{2^{d-1}}} \rceil$

The main result of this paper is the following theorem :

Theorem 4.1 There exists a set $P \subset \mathbb{R}^d$ of n points which has no monotone subsequence of length larger then $\lceil n^{\frac{1}{2^{d-1}}} \rceil$.

Let us first investigate for simplicity the case $n^{\frac{1}{2^{d-1}}} \in \mathbb{N}$. The proof for the general case works analougously and will be discussed later.

We will construct P such that the coordinates in each dimension $i \in \{1, \ldots, d\}$ are pairwise distinct, i.e. monotone subsequences of P will be monotone with respect to some order $o \in \mathcal{O}_d$. Note that in this case for two different points $p \neq q \in P$ there exists exactly one order $o \in \mathcal{O}_d$ such that $a\ o\ b$ holds.

A subsequence $[\ s_1, s_2, \ldots, s_{r-1}, s_r\ ]$ is monotone with respect to $o \in \mathcal{O}_d$ if and only if the subsequence $[\ s_r, s_{r-1}, \ldots, s_2, s_1\ ]$ is monotone with respect to $\overline{o}$. Therefore, we can restrict ourselves w.l.o.g. to the set $L_d = \{o \mid o \in \mathcal{O}_d \text{ and } o(1)\ = <\}$ of orders. Note that $\mid L_d \mid = 2^{d-1}$, $\overline{L_d} \cap L_d = \emptyset$ and $\overline{L_d} \cup L_d = \mathcal{O}_d$, where $\overline{L_d} = \{\overline{o} \mid o \in L_d\}$. Consider some order of the elements of L_d and let L_d be itself an ordered set $L_d\ =\ [\ o_1, o_2, \ldots, o_{2^{d-1}}\ ]$ of these orders.

Idea : We consider the 2^{d-1}-dimensional grid-cube $G = \{1, \ldots, n^{\frac{1}{2^{d-1}}}\}^{2^{d-1}}$ of side length $n^{\frac{1}{2^{d-1}}}$. There are n grid-points in G and we will assign to each grid-point

$$X = [x_1, x_2, \ldots, x_i, \ldots, x_{2^{d-1}}]$$

where $x_i \in \{1, \ldots, n^{\frac{1}{2^{d-1}}}\}$, exactly one point in P. With other words we define a bijective function $\Phi : G \rightarrow P$. The set $P \subset \mathbb{R}^d$ and the bijection $\Phi : G \rightarrow P$ will be defined such that the following holds: For any $p \neq q \in P$ with $p\ o_i\ q$, where $o_i \in L_d$, the inequation $x_i < y_i$ should hold, where x_i and y_i are the i-th grid-coordinate of $\Phi^{-1}(p)$ and $\Phi^{-1}(q)$, respectively:

$$\begin{aligned} \Phi^{-1}(p) &= X = [x_1, \ldots, x_i, \ldots, x_{2^{d-1}}] \\ \Phi^{-1}(q) &= Y = [y_1, \ldots, y_i, \ldots, y_{2^{d-1}}] \end{aligned}$$

Because there are at most $n^{\frac{1}{2^{d-1}}}$ paarwise distinct i-th grid-coordinates there exists no subsequence of P which is monotone with respect to o_i and has length larger than $n^{\frac{1}{2^{d-1}}}$.

Throughout this paper we consider G and L_d to be fixed. Now we present the details for the proof of Theorem 4.1.

Definition 4.1 Given are two different grid-points of $G = \{1, \ldots, n^{\frac{1}{2^{d-1}}}\}^{2^{d-1}}$

$$X = [x_1, x_2, \ldots, x_{2^{d-1}}] \text{ and } Y = [y_1, y_2, \ldots, y_{2^{d-1}}].$$

The **distinguishing index** $i_{X \neq Y} \in \{1, \ldots, 2^{d-1}\}$ of X and Y is defined as

$$i_{X \neq Y} = \min\{i \in \{1, \ldots, 2^{d-1}\} \mid x_i \neq y_i\}.$$

Definition 4.2 Let $\Phi : G \to P$ be a bijective function. We say that (P, Φ) has the **distinguishing index property** if for all $X \neq Y \in G$ with the distinguishing index $i = i_{X \neq Y}$

$$\begin{aligned} X &= [x_1, x_2, \ldots, x_{i-1}, \mathbf{x_i}, \mathbf{x_{i+1}}, \ldots, \mathbf{x_{2^{d-1}}}] \\ Y &= [x_1, x_2, \ldots, x_{i-1}, \mathbf{y_i}, \mathbf{y_{i+1}}, \ldots, \mathbf{y_{2^{d-1}}}] \end{aligned}$$

the following holds

$$x_i < y_i \implies \Phi(X)\ o_i\ \Phi(Y) \tag{2}$$
$$x_i > y_i \implies \Phi(X)\ \overline{o_i}\ \Phi(Y) \tag{3}$$

Lemma 4.1 Let (P, Φ) have the distinguishing index property. Then

$$(\ \Phi(X)\ o\ \Phi(Y) \text{ and } o = o_i \in L_d\ ) \implies (\ i = i_{X \neq Y} \text{ and } x_i < y_i\ ) \tag{4}$$
$$\left(\ \Phi(X)\ o\ \Phi(Y) \text{ and } o = \overline{o_i} \in \overline{L_d}\ \right) \implies (\ i = i_{X \neq Y} \text{ and } x_i > y_i\ ) \tag{5}$$

Proof: There is exactly one order $o \in \mathcal{O}_d$ such that $\Phi(x)\ o\ \Phi(y)$ holds. In (4) and (5) $o \in \{o_i, \overline{o_i}\}$ is given. Because (P, Φ) has the distinguishing index property, formula (2) and (3) hold, and these imply $o \in \{o_{i_{X \neq Y}}, \overline{o_{i_{X \neq Y}}}\}$. Thus,

$$\{o_i, \overline{o_i}\} \cap \{o_{i_{X \neq Y}}, \overline{o_{i_{X \neq Y}}}\} \neq \emptyset$$

holds, which implies $i = i_{X \neq Y}$.

If $o = o_i$ then $x_i < y_i$ holds, because otherwise we have $x_i > y_i$ and therefore by the distinguishing index property $o = \overline{o_i}$ holds, which is a contradiction with $o = o_i$. Formula (5) is proven analogously as above.

Q.E.D.

Lemma 4.2 Let (P, Φ) have the distinguishing index property. Then P has no monotone subsequence of length larger then $n^{\frac{1}{2^{d-1}}}$.

Proof: Let $\mathcal{S}$ be a subsequence of P of length $\mid \mathcal{S} \mid$ which is monotone with respect to some order $o \in \mathcal{O}_d$. W.l.o.g $o = o_i \in L_d$.

Let $p \neq q \in \mathcal{S}$. Then either $p\ o_i\ q$ holds or $q\ o_i\ p$. W.l.o.g. let $p\ o_i\ q$ hold. Because Φ is bijective there exists $X, Y \in G$, $X \neq Y$ with $\Phi(X) = p$ and $\Phi(Y) = q$. By Lemma 4.1 we have $i = i_{X \neq Y}$ and $x_i < y_i$. Therefore, all grid-coordinates x_i in the i-th direction of the grid-points $X = \Phi^{-1}(p)$ for all $p \in \mathcal{S}$ are pairwise distinct.

This implies

$$\mid \mathcal{S} \mid \ = \left| \left\{ x_i = \left(\Phi^{-1}(p)\right)_i \ : \ p \in \mathcal{S} \right\} \right| \ \leq \ \left| \left\{ x_i \ : \ x_i \in \{1, \ldots, n^{\frac{1}{2^{d-1}}}\} \right\} \right| \ = n^{\frac{1}{2^{d-1}}}$$

which proves the lemma. Q.E.D.

It remains to show that there exists a set $P \subset \mathbb{R}^d$ and a bijective function $\Phi : G \to P$ such that (P, Φ) has the distinguishing index property.

Let us motivate first the way we construct the set P and the function $\Phi : G \to P$.

- As the coordinates in the j-th dimension of the points in P have be chosen to be pairwise distinct, their set $P_j = \{p_j \mid p = (p_1, \ldots, p_j, \ldots, p_d) \in P\}$ can be set for simplicity to equal $\{1, \ldots, n\}$. Thus P will be choosen to be a subset of $\{1, \ldots, n\}^d$.
- In order to define the bijective function $\Phi : G \to P$ we have to define appropriate bijective functions $\Phi_j \ : \ G \ \to \ P_j = \{1, \ldots, n\}$ which set all coordinates in the j-th dimension of the points from P for $j \in \{1, \ldots, d\}$. For $\Phi : G \to P$ the following holds:

$$\Phi(X) = (\ \Phi_1(X), \ldots, \Phi_j(X), \ldots, \Phi_d(X)\ ) .$$

- Obviously, the bijection $\Phi_j^{-1} \ : \ \{1, \ldots, n\} \to G$ transfers the natural linear order on $\{1, \ldots, n\}$ to an order $<_j$ on the elements of G as follows:

$$\Phi_j^{-1}(1) \ <_j \ \Phi_j^{-1}(2) \ <_j \ \ldots \ <_j \ \Phi_j^{-1}(n)$$

  Thus, $<_j$ has the property $X \ <_j \ Y \iff \Phi_j(X) \ < \ \Phi_j(Y)$ and constructing Φ_j is is the same as constructing $<_j$.

To visualize the definition of $<_j$ consider the following table T in Figure 1 with the L_d-orders as the rows of the table. The column j of T corresponds to the j-th dimension of the d-dimensional points of P.

Definition 4.3 For $j \in \{1, \ldots, d\}$ the total order $<_j$ on G is defined as follows. For two distinct grid points $X = [x_1, \ldots, x_{2^{d-1}}]$ and $Y = [y_1, \ldots, y_{2^{d-1}}]$ of G with the distinguishing index $i = i_{X \neq Y}$ we have $X \ <_j \ Y \iff x_i\ o_i(j)\ y_i$.

Because any $X \neq Y \in G$ have a distinguishing index the order $<_j$ is a total order on G. Note that as $o_i(1) \ = < \forall i \in \{1, \ldots, 2^{d-1}\}$ the order $<_1$ on G corresponds to the lexicographic order of the elements $X = [x_1, x_2, \ldots, x_{2^{d-1}}]$ of G interpreted as strings.

Definition 4.4 Let $X_1 \ <_1 \ X_2 \ <_1 \ \ldots \ <_1 \ X_n$ be the lexicographically ordered elements of G. For any $j \in \{1, \ldots, d\}$ let $\sigma_j : \{1, \ldots, n\} \to \{1, \ldots, n\}$ be the permutation induced by the order $<_j$ defined in Definition 4.3 such that the following holds:

$$X_{\sigma_j(1)} \ <_j \ X_{\sigma_j(2)} \ <_j \ \ldots \ <_j \ X_{\sigma_j(i)} \ <_j \ \ldots \ <_j \ X_{\sigma_j(n)}.$$

Note that σ_1 is the identity permutation.

	1 2 3 ...	**j**	**..., d**
$\mathbf{o_1}$	(< , < , < , ...	$o_1(j)$	... , <)
$\mathbf{o_2}$	(< , < , < , ...	$o_2(j)$	... , >)
$\vdots$		$\vdots$	
$\mathbf{o_i}$	(< , > , < , ...	$o_i(j)$	... , >)
$\vdots$		$\vdots$	
$\mathbf{o_{2^{d-1}}}$	(< , > , > , ...	$o_{2^{d-1}}(j)$	... , >)

Figure 1: Table with the L_d-orders

Construction of set P and bijection $\Phi : G \to P$:

Let $X_1 <_1 X_2 <_1 \dots <_1 X_n$ be the lexicographically ordered elements of G. Let the bijections $\Phi_j : G \to \{1, \dots, n\}$ which set the coordinates in the j-th dimension of the points from P be defined as:

$$\Phi_j(\, X_{\sigma_j(r)} \,) \;=\; r \quad \text{for all} \quad r \in \{1, \dots, n\}$$

where σ_j is defined in Definition 4.4. Let

$$\Phi(X) = (\ \Phi_1(X), \dots, \Phi_j(X), \dots, \Phi_d(X)\ ) \quad \in \mathbb{R}^d$$

be the d-dimensional point corresponding to the grid-point $X \in G$. The set P is defined as follows :

$$P \;=\; \{\ \Phi(X_1), \dots, \Phi(X_r), \dots, \Phi(X_n)\ \}.$$

Lemma 4.3 The tuple (P, Φ) constructed above has the distinguishing index property.

Proof: Let $X \neq Y$ be some grid-points of G and let $i = i_{X \neq Y}$ be the distinguishing index of X and Y :

$$\begin{aligned} X &= [x_1, x_2, \dots, x_{i-1}, \mathbf{x_i}, \mathbf{x_{i+1}}, \dots, \mathbf{x_{2^{d-1}}}] \\ Y &= [x_1, x_2, \dots, x_{i-1}, \mathbf{y_i}, \mathbf{y_{i+1}}, \dots, \mathbf{y_{2^{d-1}}}] \end{aligned}$$

By Definition 4.2 $x_i \neq y_i$. W.l.o.g. assume $x_i < y_i$ holds. The goal is to prove that $\Phi(X)\ o_i\ \Phi(Y)$ holds, i.e. $\Phi_j(X)\ o_i(j)\ \Phi_j(Y)$ holds for all dimensions $j \in \{1, \dots, d\}$.

By the construction of Φ_j, Definition 4.3 and Definition 4.4 we have :

$$\begin{aligned} \Phi_j(X) \;<\; \Phi_j(Y) \quad &\Longleftrightarrow \quad X <_j Y \quad \Longleftrightarrow \quad x_i\ o_i(j)\ y_i \\ \Phi_j(Y) \;<\; \Phi_j(X) \quad &\Longleftrightarrow \quad Y <_j X \quad \Longleftrightarrow \quad y_i\ o_i(j)\ x_i \end{aligned}$$

Because of $x_i < y_i$ we have $\Phi_j(X)\ o_i(j)\ \Phi_j(Y)$ for all $j \in \{1, \dots, d\}$.

Q.E.D.

In the following we illustrate the construction of the point set P for the case $\mathbf{d = 3}, \mathbf{n = 3^4}$.

Example 4.1 Consider the case $\mathbf{d = 3, n = 3^4} = 81$. Because of $2^{d-1} = 4$ and $n^{\frac{1}{2^{d-1}}} = 3$ the grid-cube G is 4-dimensional, has length 3 and equals $\{1,2,3\}^4$. Let the list L_3 which we illustrate in Figure 2 be defined as $L_3 = [\ (<,<,<); (<,<,>); (<,>,<); (<,>,>)\ ]$.

	1	**2**	**3**
$\mathbf{o_1}$	(< ,	<	, <)
$\mathbf{o_2}$	(< ,	<	, >)
$\mathbf{o_3}$	(< ,	>	, <)
$\mathbf{o_4}$	(< ,	>	, >)

Figure 2: Table with the L_3-orders

Figure 3 shows the lexicographically ordered grid-points $[x_1, x_2, x_3, x_4]$ of G, where $x_i \in \{1,2,3\}$. We have $X_1 = [1,1,1,1], X_2 = [1,1,1,2], \ldots, X_9 = [1,1,3,3]$ and so on till $X_{81} = [3,3,3,3]$. The arrow labeled by $i \in \{1,2,3,4\}$ indicates the direction in which the i-th grid-coordinate grows. P and Φ should be defined appropriately such that (P, Φ) has the

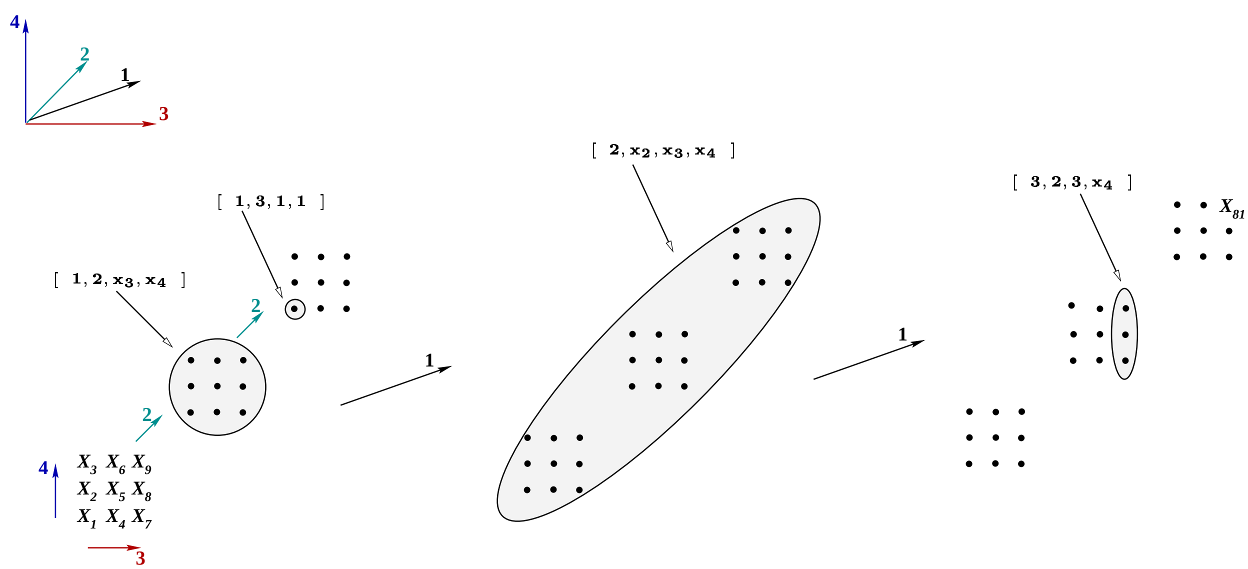

Figure 3: The 4-dimensional grid-cube G of sidelength 3 in the case $d = 3$ and $n = 3^4$

distinguishing index property. The bijection $\Phi_j : G \to \{1,2,\ldots,81\}$ ($j \in \{1,2,3\}$) sets the coordinates in the j-th dimension of the points in P. $\Phi_1 : G \to \{1,2,\ldots,81\}$ is defined such that $\Phi_1(X_r) = r$ holds for all $r \in \{1,2,\ldots,81\}$.

For illustration we show now how to set the coordinates in the 2-nd dimension of the points in P, with other words we define $\Phi_2 : G \to \{1,2,\ldots,81\}$. For this we consider two distinct grid-points $X = [x_1, x_2, x_3, x_4]$ and $Y = [y_1, y_2, y_3, y_4]$ of G. In order to have $\Phi_2(X) < \Phi_2(Y)$ one of the following cases should occur :

a) $i_{X \neq Y} = 1$ and $x_1 < y_1$ (because $o_1(2) \ = <$)

b) $i_{X \neq Y} = 2$ and $x_2 < y_2$ (because $o_2(2) \ = <$)

c) $i_{X \neq Y} = 3$ and $x_3 > y_3$ (because $o_3(2) \ = >$)

d) $i_{X \neq Y} = 4$ and $x_4 > y_4$ (because $o_4(2) \ = >$)

This implies because of $X <_2 Y \iff \Phi_2(X) < \Phi_2(Y)$ the definition of the order $<_2$ on G and of the bijective function $\Phi_2 : G \to P_2 = \{1, 2, \ldots, 81\}$ which is illustrated in Figure 4. T_2 is the second column of the table with the L_3-orders in Figure 2. We have $\Phi_2(X_1) = 9, \Phi_2(X_2) = 8, \ldots, \Phi_2(X_9) = 1$ and so on till $\Phi_2(X_{81}) = 73$.

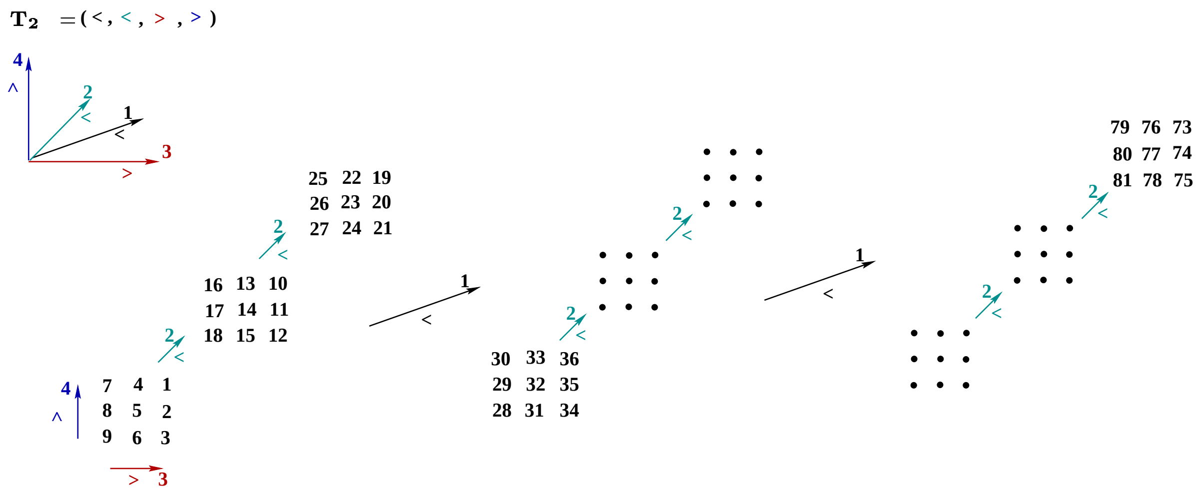

Figure 4: Setting the coordinates in the 2-nd dimension of the points in P

Analogously we define $\Phi_3 : G \to P_3 = \{1, 2, \ldots, 81\}$ which sets the coordinates in the 3-rd dimension of the points in P.

P is given by : $P = \{ \ (\ 1, 9, 21\ ), (\ 2, 8, 20\ ), \ldots, (\ r, \Phi_2(r), \Phi_3(r)\ ), \ldots, (\ 81, 73, 61\ )\ \}$.

Now, Lemma 4.2 and Lemma 4.3 imply Theorem 4.1 for the case $n^{\frac{1}{2^{d-1}}} \in \mathbb{N}$. The general case works as follows. Let $m = \lceil n^{\frac{1}{2^{d-1}}} \rceil$. Thus, $n \leq m^{2^{d-1}}$ holds. Now let P_m be a set of $m^{2^{d-1}}$ points in $\mathbb{R}^d$ with longest monotone subsequence of length at most m, and which is constructed as discussed above. Take any subset $P \subset P_m$ of $n \leq m^{2^{d-1}}$ points. P has also no monotone subsequence of length larger then $m = \lceil n^{\frac{1}{2^{d-1}}} \rceil$. This completes the proof of Theorem 4.1.

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