Least Upper Bounds for the Size of OBDDs Using Symmetry Properties

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Abstract—This paper investigates reduced ordered binary decision diagrams (OBDD) of partially symmetric Boolean functions when using variable orders where symmetric variables are adjacent. We prove upper bounds for the size of such symmetry ordered OBDDs (SymOBDD). They generalize the upper bounds for the size of OBDDs of totally symmetric Boolean functions and nonsymmetric Boolean functions proven by Heap and Mercer [14], [15] and Wegener [37]. Experimental results based on these upper bounds show that the nontrivial symmetry sets of a Boolean function should be located either right up at the beginning or right up at the end of the variable order in order to obtain best upper bounds.

Index Terms—Binary decision diagrams, variable ordering, upper worst case bounds, partial symmetric Boolean functions.

1 Introduction

Binary Decision Diagrams (BDDs) as a data structure for representation of Boolean functions were first introduced by Lee [24] and further popularized by Akers [1] and Moret [30]. In the restricted form of reduced ordered BDDs (OBDD) they gained widespread application because OBDDs are a canonical representation and allow efficient manipulations [3]. Some fields of application are logic design verification, test generation, fault simulation, and logic synthesis [4], [5], [7], [9], [13], [16].

Most of the algorithms using OBDDs have runtime polynomial in the size of the OBDDs. The sizes themselves depend on the variable order used. Thus, there is a need to find a variable order that minimizes the number of nodes in an OBDD. In particular, the importance of this problem is demonstrated by the large number of projects working on heuristics for finding good variable orders [11], [12], [19], [27], [33], [34].

In this paper, we consider partially symmetric Boolean functions, i.e., Boolean functions that are invariant under the permutation of some input variables. Knowing a Boolean function to be partially symmetric allows application of special logic synthesis tools that can improve the results of the design [8], [10], [21], [32], [34]. Furthermore, knowing the variables of a Boolean function which are symmetric often restricts the search space of a logic design problem which may yield a remarkable decrease of running time for that problem. Such problems are, e.g., permutation independent Boolean comparison [6], [23], [28], [29] and technology mapping [26]. Note that very efficient algorithms are known in literature which determine symmetries for both completely and incompletely specified Boolean functions [34].

Jeong et al. [20] have empirically observed that symmetric variables tend to be adjacent in optimum orders. This has been experimentally proven to be the case for most Boolean functions with up to five variables [34]. The value of a function that is symmetric in some variables $\{x_{i_1}, \ldots, x_{i_q}\}$ does not depend on the exact assignment of these variables, but only on their weight $\sum_{j=1}^{q} x_{i_j}$. If one uses OBDDs based on variable orders in which symmetric variables are adjacent, the weight of symmetric variables is computed in neighboring levels and no information about partial weights has to be kept over several nonsymmetric levels. This leads to a special class of variable orders where the symmetric variables are located side by side and are treated as fixed blocks. We call these orders *symmetry orders*. The corresponding OBDDs are called *symmetry ordered OBDDs* and are denoted by SymOBDD.

To give an impressive example for the fact that it helps to locate the symmetric variables side by side, consider the function $x_1x_{n+1} + \ldots + x_nx_{2n}$ [3]. The size of the corresponding OBDD with variable order $x_1, x_2, \ldots, x_{2n}$ is exponential in n, whereas the size of any SymOBDD of the function is linear in n. However, note that, although it is reasonable to locate the symmetric variables side by side, it does not lead to optimal results in all cases. A counterexample has been given in [31] showing a linear gap between optimal orders and best symmetry orders.

Recently, some theoretical results have been proven which justify the restriction to SymOBDDs [35]. The first result concerns Boolean functions with symmetry sets of very different sizes. It states that, although almost all Boolean functions with that property have no symmetry order which is optimal, it is reasonable to locate symmetric variables side by side as the performance ratio for these Boolean functions randomly chosen according to the uniform distribution is smaller than $1 + \delta$ with probability close to 1 (for all constant $\delta > 0$). The performance ratio for function f is defined as the ratio between the size of a best SymOBDD of f and the size of an optimal OBDD of f. In the

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case of Boolean functions which have symmetry sets of nearly equal sizes, it has been even proven that the performance ratio is 1 with probability close to 1.

In this paper, we investigate SymOBDDs of Boolean functions (even those which are not partially symmetric in any pair of variables at all). We derive upper bounds for the size of the SymOBDDs that only depend on the size of the maximal blocks of pairwise symmetric variables which we call *symmetry sets* in the following. These results generalize the upper bounds for totally symmetric Boolean functions [14], [37], nonsymmetric Boolean functions [15], and Boolean functions with one symmetry set [25].

Experiments based on the upper bounds proven show that the largest symmetry set of a Boolean function should be located right up at the beginning of the variable order in most cases. In any case, the largest symmetry set of a Boolean function should be located either right up at the beginning or at the end of the variable order. The remaining nontrivial symmetry sets of a Boolean function should be put next to the largest symmetry set or right up at the other side of the variable order in order to guarantee best performance compared to the other SymOBDDs with respect to the upper bounds.

This paper is structured as follows: In Section 2, we review some basic notations and definitions. The upper bounds for the size of SymOBDDs are proven in Sections 3, 4, and 5. The open problem which has to be solved in order to show the tightness of the upper bounds is given in Section 6. Finally, we present experimental results in Section 7.

2 Preliminaries

In this section, we review basic notations and definitions that are needed for the understanding of the paper.

2.1 Partially Symmetric Boolean Functions

Let $f : \{0,1\}^n \rightarrow \{0,1\}$ be a completely specified Boolean function and $X = \{x_1, \ldots, x_n\}$ be the corresponding set of variables. The function f is said to be *symmetric* with respect to a set $\lambda \subseteq X$ if f remains invariant under all permutations of the variables in λ. For completely specified Boolean functions, symmetry is an equivalence relation which partitions the set X into disjoint classes $\lambda_1, \ldots, \lambda_k$ that will be named the *symmetry sets*. A function f is called *partially symmetric* if it has at least one symmetry set λ_i with size $|\lambda_i| > 1$. If a function f has only one symmetry set $\lambda = X$, then f is called *totally symmetric*.

Sometimes, it may occur that $\{x_i, x_j\}$ is not a symmetric pair, but $\{x_i, \overline{x}_j\}$ is one. This kind of symmetry was introduced by Hurst [18] and named *equivalence symmetry*. In such a case, the phase of variable x_j can be changed to get symmetry in $\{x_i, x_j\}$.

2.2 Binary Decision Diagrams

We start with a brief review of the essential definitions and properties of *Binary Decision Diagrams*(BDDs) as introduced in [3]. As we need binary decision diagrams only to represent Boolean functions in this paper, we define them directly over a set of variables.

Definition 1. *A* Binary Decision Diagram (BDD) *is a rooted directed acyclic graph $G = (V, E)$ with vertex set V containing two types of vertices,* nonterminal *and* terminal *vertices. A nonterminal vertex v has as a label a variable $index(v) \in \{x_1, \ldots, x_n\}$ and two children $low(v), high(v) \in V$. A terminal vertex v is labeled with a value $value(v) \in \{0,1\}$ and has no outgoing edge.*

A BDD having root vertex v denotes a Boolean function f_v which is the constant function $value(v)$ if v is a terminal vertex and which is defined by $\overline{x}_i \cdot f_{low(v)}(x_1, \ldots, x_n) + x_i \cdot f_{high(v)}(x_1, \ldots, x_n)$ if v is a nonterminal vertex with $index(v) = x_i$. In this case, the variable x_i is called the decision variable for v.

In order to obtain a canonical representation of Boolean functions, reduced ordered BDDs have to be considered.

Definition 2. *A* Reduced Ordered BDD (OBDD) *is a BDD which is ordered, i.e., there is a fixed bijective function $\pi : \{1, \ldots, n\} \rightarrow \{x_1, \ldots, x_n\}$,* the variable order *of the OBDD, such that, for any two nonterminal vertices v and w for which there is a directed path from v to w, the inequation $\pi^{-1}(index(v)) < \pi^{-1}(index(w))$ holds, and which is reduced, i.e., there exists no $v \in V$ with $low(v) = high(v)$ and there are no two vertices v and v' such that the sub-BDDs rooted by v and v' are isomorphic.*

Definition 3. *Let f be a Boolean function with the symmetry sets $\{\lambda_1, \ldots, \lambda_k\}$. A variable order π is called a* symmetry variable order *if, for each symmetry set λ_i, there exists j so that $\{\pi(j), \pi(j+1), \ldots, \pi(j + |\lambda_i| - 1)\} = \lambda_i$.*

By this definition, the class of symmetry variable orders consists of all variable orders where the variables of each symmetry set are located side by side. The OBDDs that correspond to symmetry orders are called *symmetry ordered* OBDDs (SymOBDD).

3 A General Upper Bound for SymOBDDs

In this section, we prove upper bounds on the size of SymOBDDs. We derive an upper bound on the number of vertices at each level, add these bounds over all levels, and get an upper bound on the number of nonterminal vertices of the considered SymOBDD.

We concentrate on Boolean functions f in $n \geq 3$ variables. Let $\lambda_1, \ldots, \lambda_m$ denote the symmetry sets of f, where $m \geq 1$. Let $k_1, \ldots, k_m \geq 1$ denote the sizes of these sets, respectively. The symmetry sets of size 1 are called trivial symmetry sets. Note that $\sum_{i=1}^{m} k_i = n$ holds. Without loss of generality, we assume that the variable order π_{sym} of the SymOBDD is fixed to $x_1, \ldots, x_n$ and that the symmetry set λ_i is located at the positions $s_i, \ldots, s_i + k_i - 1$ with $s_i = 1 + \sum_{j=1}^{i-1} k_j$, $\forall i \in \{1, \ldots, m\}$.

The crucial idea is that the maximum number of cofactors with respect to $x_1, \ldots, x_{q-1}$ of the Boolean function f is an upper bound on the number of vertices at level $q \in \{1, \ldots, n\}$ in an OBDD of the function f. Another upper bound on the number of vertices at level q is the number of Boolean functions defined on $\{x_q, \ldots, x_n\}$ and which depend on x_q. We investigate these two upper bounds on the number of vertices at a certain level q in the following lemma. Let us first introduce the following notations:

$$Precede(u) = \prod_{j=1}^{u}(k_j + 1) \quad \forall 1 \le u \le m$$
$$Precede(0) = 1$$
$$Follow(u) = \prod_{j=u}^{m}(k_j + 1) \quad \forall 1 \le u \le m$$
$$Follow(m + 1) = 1.$$

$Precede(u)$ and $Follow(u)$ represent the number of assignments with different weights of the variables in the symmetry sets $\lambda_1, \dots, \lambda_u$ and in the symmetry sets $\lambda_u, \dots, \lambda_n$, respectively. These values are of interest with respect to the number of different partially symmetric Boolean functions and the number of different cofactors because the output of the function f does not depend on the exact assignment of the variables in the symmetry sets λ_i, but only on their weights $\sum_{x_j \in \lambda_i} x_j$.

Lemma 1. *The maximum number of nonterminal vertices of a SymOBDD of a Boolean function with the properties just described is given by* $\sum_{q=1}^{n} W_q$ *with* $W_q \le F(q) := min\{G(q), H(q)\}$, *where G and H are defined by:*

$$G(q) = \begin{cases} q & if\ 1 \le q \le k_1 + 1 \\ F(s_i) \cdot (q - s_i + 1) & if\ s_i < q \le s_i + k_i \\ & for\ some\ i \ge 2. \end{cases}$$

and

$$H(q) = 2^{Follow(i+1)(s_i+k_i-q+1)} - 2^{Follow(i+1)}$$

with $s_i \le q \le s_i + k_i - 1$ *for some* $i \ge 1$, *respectively.*

Proof. We prove the statement in two steps.

- First, we show that $H(q)$ gives the number of Boolean functions defined on $\{x_q, \dots, x_n\}$ which have $\{x_q, \dots, x_{s_i+k_i-1}\}, \lambda_{i+1}, \lambda_{i+2}, \dots, \lambda_m$ as symmetry sets and which depend on x_q. With this, it follows that there are at most $H(q)$ vertices with label x_q in an OBDD.

  There are $(s_i + k_i - 1 - q + 1 + 1) \cdot (|\lambda_{i+1}| + 1) \cdot \dots \cdot (|\lambda_m| + 1)$ assignments with different weights of the symmetry sets. Thus, there are $2^{Follow(i+1)(s_i+k_i-q+1)}$ different Boolean functions with the above symmetry sets. $2^{Follow(i+1)}$ of them do not depend on x_q (they also do not depend on $x_{q+1}, \dots, x_{s_i+k_i-1}$ because of symmetry). Therefore, there are at most $H(q)$ vertices in an OBDD with label x_q.
- Now, we show by induction on q that $W_q \le G(q)$ holds $\forall q$. The idea is that $G(q)$ gives an upper bound on the number of the cofactors with respect to $x_1, \dots, x_{q-1}$ of the Boolean function f considered, which obviously is an upper bound on the number of vertices with label x_q.

  First, consider $q \le k_1 + 1$. As $x_1, \dots, x_{q-1}$ belong to the same symmetric set and there are at most q different weights for this variable subset, there are at most q different cofactors with respect to $x_1, \dots, x_{q-1}$ of f. This proves $W_q \le q =: G(q)$ for $q \le k_1 + 1$. Thus, $W_q \le F(q)$ for $q \le k_1 + 1$.

  Now, consider q with $s_i < q \le s_i + k_i$ for some $i \ge 2$. (Note that $s_{i+1} = s_i + k_i$ holds.) As $x_{s_i}, \dots, x_{q-1}$ belong to the same symmetry set and there are at most $q - s_i + 1$ different weights for this subset of variables, there are at most $q - s_i + 1$ different cofactors with respect to $x_{s_i}, \dots, x_{q-1}$ of any of the vertices of level s_i. Thus, there are at most $F(s_i) \cdot (q - s_i + 1)$ cofactors with respect to $x_1, \dots, x_{q-1}$. This proves $W_q \le G(q)$ and, so, $W_q \le F(q)$. □

In Lemma 1, the upper bound $F(q)$, $1 \le q \le n$, has been defined recursively. The following lemma gives a solution to this recurrence.

Lemma 2. *Let* $g(q)$ *be defined as* $Precede(i - 1) \cdot (q - s_i + 1)$ *for* $s_i \le q \le s_i + k_i - 1$, $i \in \{1, \dots, m\}$. *Then, the equation* $F(q) = min\{g(q), H(q)\}$ *holds for all* $q \in \{1, \dots, n\}$.

Proof. We prove the statement by induction on q.

Because of $Precede(0) = 1$ and $s_1 = 1$, the equation $g(q) = G(q)$ holds for $1 \le q \le k_1$. Furthermore, $g(s_2) := Precede(1) = k_1 + 1 = G(k_1 + 1) =: G(s_2)$ also holds. Thus, the equations $g(q) = G(q)$ and $F(q) = min\{g(q), H(q)\}$ hold for all $q \le s_2$.

Now, consider q with $s_i < q \le s_i + k_i$ for some $i \ge 2$. By definition (see Lemma 1), $F(q) := min\{G(q), H(q)\}$ and $G(q) := F(s_i) \cdot (q - s_i + 1)$. Applying the induction assumption, we obtain the equation

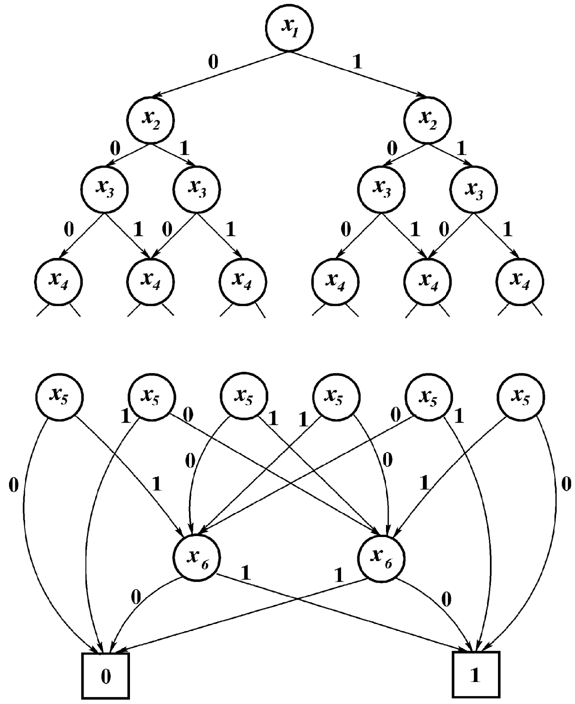
$$G(q) = min\{g(s_i), H(s_i)\}(q - s_i + 1).$$

Assuming $g(s_i) \le H(s_i)$ implies $G(q) = g(s_i)(q - s_i + 1) = Precede(i - 1)(q - s_i + 1) =: g(q)$ which immediately proves $F(q) = \min\{g(q), H(q)\}$.

Assuming $H(s_i) \le g(s_i)$ implies $F(q) = min\{H(s_i)(q - s_i + 1), H(q)\}$ and $H(q) < g(q)$ as g is strictly increasing and H strictly decreasing. The monotony of H also implies $H(s_i) > H(q)$ and $H(s_i)(q - s_i + 1) > H(q)$. Thus, the equation $F(q) = H(q)$ holds. This proves $F(q) = H(q) = min\{g(q), H(q)\}$. □

Consider $q \in \{s_i, \dots, s_i + k_i - 1\}$. The value $g(q)$ represents the number of assignments with different weights of the variables in the symmetry sets $\lambda_1, \dots, \lambda_{i-1}, \{x_{s_i}, \dots, x_{q-1}\}$, which is a bound on the maximum number of cofactors with respect to $x_1, \dots, x_{q-1}$ of the Boolean function f. The value $H(q)$ is the number of Boolean functions defined on $\{x_q, \dots, x_n\}$ which have $\{x_q, \dots, x_{s_i+k_i-1}\}$, $\lambda_{i+1}, \dots, \lambda_m$ as symmetry sets and which depend on x_q. The two upper bounds $g(q)$ and $H(q)$ are illustrated by the following example.

Example 1. Consider a function $g : \{0,1\}^6 \to \{0,1\}$ in the variables $x_1, x_2, \dots, x_6$ with the following symmetry sets: $\lambda_1 = \{x_1\}, \lambda_2 = \{x_2, x_3\}$, and $\lambda_3 = \{x_4, x_5, x_6\}$. Fig. 1 shows the largest SymOBDD which has the properties of function



Level q	$F(q)$	$g(q)$	$H(q)$
1	$1 = g(1)$	1	$2^{24} - 2^{12}$
2	$2 = g(2)$	2	$2^{12} - 2^{4}$
3	$4 = g(3)$	$2 \cdot 2$	$2^{8} - 2^{4}$
4	$6 = g(4)$	$2 \cdot 3$	$2^{4} - 2$
5	$6 = H(5)$	$6 \cdot 2$	$2^{3} - 2$
6	$2 = H(6)$	$6 \cdot 3$	$2^{2} - 2$

Fig. 1. Maximum number of nodes at the levels of function g.

g. The table shows the values of $F(q) = min\{g(q), H(q)\}$ for the various levels q of such a SymOBDD.

At level 4, there are at most 6 nodes, as the cofactor $g_{x_1=b,x_2=0,x_3=1}$ with $b \in \{0,1\}$ must be equal to the cofactor $g_{x_1=b,x_2=1,x_3=0}$ because $\{x_2, x_3\}$ is a symmetry set. Up to level 4, the upper bound is given by $g(q)$. For the rest of the levels (5 and 6), the upper bound is given by $H(q)$.

Now, let us investigate the evaluation of

$$F(q) = min\{g(q), H(q)\},$$

for $q \in \{1, \ldots, n\}$.

Lemma 3. *There is a unique turnpoint t so that*

$$F(q) = \begin{cases} g(q) & if\ 1 \leq q \leq t \\ H(q) & if\ t < q \leq n \end{cases}$$

holds for $n \geq 3$.

Proof. By Lemma 2, we have $F(q) = min\{g(q), H(q)\}$ for all $q \in \{1, \ldots, n\}$. It is obvious that as g is strictly increasing, H is strictly decreasing. Furthermore, the inequations $g(1) < H(1)$ and $g(n) > H(n)$ are satisfied because

$$g(1) = 1 < H(1) = 2^{Follow(2)} \cdot \left(2^{k_1+1} - 2\right)$$

$$H(n) = 2 < g(n) = k_m \cdot \prod_{j=1}^{m-1} (k_j + 1)$$

hold for $n \geq 3$. This completes the proof. □

Theorem 1. *Let f be a Boolean function in $n \geq 3$ variables with the symmetry sets $\lambda_1, \ldots, \lambda_m$ of sizes $k_1, \ldots, k_m \geq 1$, respectively. The upper bound on the size of the SymOBDD with the variable order π_{sym} is given by*

$$\sum_{q=1}^{t} g(q) + \sum_{q=t+1}^{n} H(q).$$

If there exists i with $g(s_i) \leq H(s_i)$ and

$$g(s_i + k_i - 1) > H(s_i + k_i - 1),$$

then turnpoint t equals

$$t_\delta := s_i + k_i - 1 - \left\lfloor \frac{\log_2\{E_1 \cdot [E_2 \cdot (k_i + 1) - \log_2 E_1 + 1]\}}{E_2} \right\rfloor + \delta$$

for some $\delta \in \{0, 1, 2\}$ with $E_1 = Precede(i - 1)$ and $E_2 = Follow(i + 1)$.

Proof. By definition, for all q with $s_i \leq q \leq s_i + k_i - 1$, the equations

$$g(q) = E_1(q - s_i + 1),$$
$$H(q) = 2^{E_2(s_i + k_i - q + 1)} - 2^{E_2},$$
$$E_1 \geq 1,$$

and $E_2 \geq 1$ hold. To prove the theorem, we show that $g(t_3) > H(t_3)$ and $g(t_0) \leq H(t_0)$ hold.

Note that $E_2 \cdot (k_i + 1) - \log_2 E_1 + 1 \geq 1$ is implied by $g(s_i) \leq H(s_i)$, i.e., $E_1 \leq 2^{E_2(k_i+1)} - 2^{E_2}$. Thus, the *log*-term in the expression of t_δ is nonnegative.

Now, we prove $g(t_0) \leq H(t_0)$. For this, we use $\phi(E_1, E_2, k_i)$ to abbreviate the formula

$$\log_2\{E_1 \cdot [E_2 \cdot (k_i + 1) - \log_2 E_1 + 1]\}.$$

The following equations hold:

$$\begin{aligned} g(t_0) &= E_1 \cdot \left(k_i - \left\lfloor \frac{\phi(E_1, E_2, k_i)}{E_2} \right\rfloor\right) \\ &\leq E_1 \cdot \frac{E_2 \cdot (k_i + 1) - \phi(E_1, E_2, k_i)}{E_2} \\ &\leq E_1 \cdot \frac{E_2 \cdot (k_i + 1) - \log_2 E_1}{E_2} \\ &\leq 2^{E_2} \cdot E_1 \cdot (E_2(k_i + 1) - \log_2 E_1) \\ &\leq 2^{E_2} \cdot E_1 \cdot (E_2(k_i + 1) - \log_2 E_1) + \\ &\quad 2^{E_2} \cdot E_1 - 2^{E_2} \\ &= 2^{E_2} \cdot E_1 \cdot (E_2(k_i + 1) - \log_2 E_1 + 1) - 2^{E_2} \\ &= 2^{E_2 \cdot \left(\frac{\log_2\{E_1 \cdot [E_2 \cdot (k_i+1) - \log_2 E_1 + 1]\}}{E_2} + 1\right)} - 2^{E_2} \\ &< 2^{E_2 \cdot \left(\left\lfloor \frac{\log_2\{E_1 \cdot [E_2 \cdot (k_i+1) - \log_2 E_1 + 1]\}}{E_2} \right\rfloor + 2\right)} - 2^{E_2} \\ &= H(t_0). \end{aligned}$$

To close the proof, we show $g(t_3) > H(t_3)$.

For that, let us give an upper bound for $\left\lfloor \frac{\phi(E_1, E_2, k_i)}{E_2} \right\rfloor$, first. Because of $\log_2(x) \leq \frac{x}{2} + 1$, we have

$$
\begin{aligned}
&\log_2(E_2 \cdot (k_1 + 1) - \log_2 E_1 + 1) \\
&\quad \leq \frac{E_2 \cdot (k_i + 1) - \log_2 E_1 + 1}{2} + 1 \\
&\Rightarrow \log_2 E_1 + \log_2(E_2 \cdot (k_i + 1) \log_2 E_1 + 1) \\
&\quad \leq \frac{E_2 \cdot (k_i + 1)}{2} + \frac{\log_2 E_1}{2} + \frac{3}{2} \\
&\Rightarrow \frac{\log_2\{E_1 \cdot [E_2 \cdot (k_i + 1) - \log_2 E_1 + 1]\}}{E_2} \\
&\quad \leq \frac{k_i + 1}{2} + \frac{\log_2 E_1}{2E_2} + \frac{3}{2E_2}.
\end{aligned}
$$

By this, it follows that

$$
\begin{aligned}
g(t_3) &= E_1 \cdot \left(k_i - \left\lfloor \frac{\phi(E_1, E_2, k_i)}{E_2} \right\rfloor + 3 \right) \\
&= E_1 \cdot \left(k_i + 1 - \left\lfloor \frac{\phi(E_1, E_2, k_i)}{E_2} \right\rfloor + 2 \right) \\
&\geq E_1 \cdot \left(k_i + 1 - \frac{k_i + 1}{2} - \frac{\log_2 E_1 + 3}{2E_2} + 2 \right) \\
&= E_1 \cdot \left(\frac{k_i + 1}{2} - \frac{\log_2 E_1}{2E_2} + \frac{4E_2 - 3}{2E_2} \right) \\
&\geq \frac{E_1 \cdot (E_2 \cdot (k_i + 1) - \log_2 E_1 + 1)}{2E_2} \\
&> \frac{E_1 \cdot (E_2 \cdot (k_i + 1) - \log_2 E_1 + 1)}{2^{E_2}} - 2^{E_2} \\
&= 2^{E_2 \cdot \left(\frac{\phi(E_1, E_2, k_i)}{E_2} - 1 \right)} - 2^{E_2} \\
&\geq 2^{E_2 \cdot \left(\left\lfloor \frac{\phi(E_1, E_2, k_i)}{E_2} \right\rfloor - 1 \right)} - 2^{E_2} \\
&= H(t_3)
\end{aligned}
$$

holds. □

4 Upper Bound in the Case of Exactly One Nontrivial Symmetry Set

In the following, we concentrate on Boolean functions $f : \{0,1\}^n \to \{0,1\}$ with $n-k$ symmetry sets of size 1 and one symmetry set λ of size $k \geq 1$. Let symmetry set λ be located at the positions $s, \ldots, s+k-1$. By the lemmas of Section 3, we easily conclude:

Corollary 1. *The maximum number of nonterminal vertices of a SymOBDD of a partially symmetric Boolean function with the properties just described is given by* $\sum_{q=1}^{n} W_q$ *with* $W_q \leq min\{g(q), H(q)\}$*, where* g *and* H *are defined as follows:*

$$
g(q) = \begin{cases}
2^{q-1} & for\ 1 \leq q \leq s-1 \\
2^{s-1} \cdot (q - s + 1) & for\ s \leq q \leq s+k-1 \\
2^{q-k-1} \cdot (k+1) & for\ s+k \leq q \leq n,
\end{cases}
$$

$$
H(q) = \begin{cases}
2^{(k+1) \cdot 2^{n-k-q+1}} - 2^{(k+1) \cdot 2^{n-k-q}} \\
\qquad for\ 1 \leq q \leq s-1 \\
2^{(s+k-q+1) \cdot 2^{n-s-k+1}} - 2^{2^{n-s-k+1}} \\
\qquad for\ s \leq q \leq s+k-1 \\
2^{2^{n-q+1}} - 2^{2^{n-q}} \\
\qquad for\ s+k \leq q \leq n.
\end{cases}
$$

The upper bound depends on the number n of variables, on the size k and on the position s of the symmetry set λ. It is given by:

$$
bound(n, k, s) = \sum_{q=1}^{t} g(q) + \sum_{q=t+1}^{n} H(q).
$$

With Corollary 1, we can specify the location of the turnpoint t more exactly than in Theorem 1.

Theorem 2.

- *If* $2^{s-1 \cdot (k+1) \leq 2^{2^{n-s-k+1}} - 2^{2^{n-s-k}}}$*, then turnpoint* t *is behind the location of* λ *and equals* $t_\delta := n - \lfloor \log_2(n - k + \log_2(k+1)) \rfloor + \delta$ *for some* $\delta \in \{0, 1\}$.
- *If*

$$
\begin{aligned}
&2^{2^{n-k-s+1}} - 2^{2^{n-k-s}} < 2^{s-1} \cdot (k+1) \\
&\quad \leq (k+1) \cdot \left(2^{(k+1) \cdot 2^{n-k-s+1}} - 2^{2^{n-k-s+1}} \right)
\end{aligned}
$$

holds, then turnpoint t *is located inside the location of* λ *and equals*

$$
\begin{aligned}
t_\delta &:= k + s - 1 \\
&\quad - \left\lfloor \frac{\log_2\left(2^{n-k} \cdot (k+1) - 2^{s-1}(s-2)\right)}{2^{n-s-k+1}} \right\rfloor + \delta
\end{aligned}
$$

for some $\delta \in \{0, 1, 2\}$.

- *If* $2^{s-1} > 2^{(k+1) \cdot 2^{n-k-s+1}} - 2^{2^{n-k-s+1}}$*, then turnpoint* t *is ahead of symmetry set* λ *and equals*

$$
t_\delta := n - k - \left\lfloor \log_2\left(\frac{n - k + \log_2(k+1)}{k+1} \right) \right\rfloor + \delta
$$

for some $\delta \in \{0, 1\}$.

The detailed proof can be found in [17].

Theorem 2 generalizes the results shown by [14], [15], [37].

Corollary 2.

- *For totally symmetric Boolean functions, turnpoint* t *with respect to the upper bound is given by* $n - \lfloor \log_2(n+2) \rfloor + \delta$ *for some* $\delta \in \{0, 1, 2\}$.
- *For Boolean functions which are not symmetric in any pair of variables, turnpoint* t *with respect to the upper*

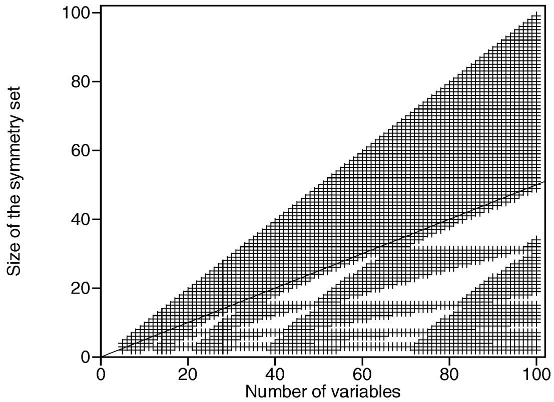


Fig. 2. Areas in which fore SymOBDDs have minimal upper bounds.

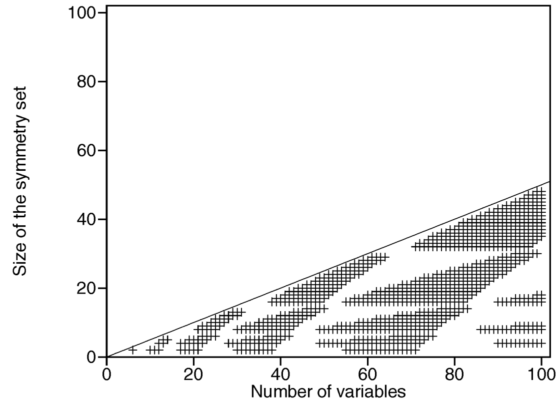


Fig. 3. Areas in which back SymOBDDs have minimal upper bounds.

bound is given by $n - \lfloor \log_2 n \rfloor + \delta$ *for some* $\delta \in \{0, 1\}$.

Proof. For totally symmetric Boolean functions, $k = n$ and $s = 1$ hold. This implies Statement 1. For Boolean functions not symmetric in any pair of variables, $k = 1$ holds. Without loss of generality, we can assume $s = 1$. This implies Statement 2. □

5 Upper Bounds for Fore and Back SymOBDDs

Now, let us concentrate on the case of exactly one nontrivial symmetry set located either right up at the beginning or right up at the end of the variable order, i.e., $1 \leq k \leq n$ and $s = 1$ or $s = n - k + 1$.

Definition 4. *Symmetry orders in which the nontrivial symmetry block is located at position* $1, \ldots, k$ *are called* fore symmetry orders. *Symmetry orders in which the nontrivial symmetry block is located at position* $n - k + 1, \ldots, n$ *are called* back symmetry orders.

By Theorem 2, we directly obtain the following two corollaries:

Corollary 3. *If we only consider fore SymOBDDs, then the following statements on turnpoint* t *hold:*

- *If* $k + 1 \leq 2^{2^{n-k}} - 2^{2^{n-k-1}}$, *then turnpoint* t *is behind position* k *and equals*
$$n - \lfloor \log_2 (n - k + \log_2 (k + 1) \rfloor + \delta$$
for some $\delta \in \{0, 1\}$.
- *If* $k + 1 > 2^{2^{n-k}} - 2^{2^{n-k-1}}$, *then turnpoint* t *is ahead position* $k + 1$ *and equals*
$$k - \left\lfloor \frac{\log_2 \left(2^{n-k} \cdot (k + 1) + 1\right)}{2^{n-k}} \right\rfloor + \delta$$
for some $\delta \in \{0, 1, 2\}$.

Corollary 4. *If we only consider back SymOBDDs, then the following statements on turnpoint* t *hold:*

- *If* $2^{n-k} > 2^{k+1} - 2$, *then turnpoint* t *is ahead position* $n - k + 1$ *and equals*
$$n - k - \left\lfloor \log_2 \left(\frac{n - k + \log_2 (k + 1)}{k + 1} \right) \right\rfloor + \delta$$
for some $\delta \in \{0, 1\}$.
- *If* $2^{n-k} \leq 2^{k+1} - 2$, *then turnpoint* t *is behind position* $n - k$ *and equals* $n - \lfloor \log_2 (2^{n-k} \cdot (2k - n + 2)) \rfloor + \delta$ *for some* $\delta \in \{0, 1, 2\}$.[1]

6 Open Problem

As already mentioned, the upper bounds proven here cover the upper bounds for nonsymmetric and totally symmetric Boolean functions proven by [14], [15], [37].

The upper bound for nonsymmetric Boolean functions has been proven to be tight in [15]. Unfortunately, the proof presented in [14] that the upper bound for totally symmetric Boolean functions is tight is faulty. We have presented a counterexample for the proof of the crucial Lemma 2 of Heap in [17, p. 64]. However, the upper bound for totally symmetric Boolean functions has been proven to be tight when the number n of variables equals $2^k + k - 2$ for some k (see [37, Theorem 7]).

The question as to whether the upper bounds proven here are tight in any case is an open problem. For the case of exactly one nontrivial symmetry set, we have proven in [17] that the tightness of the upper bounds would be implied by a positive answer of the question as to whether for any *de Bruijn graph* [2, pp. 236-237], there is a Eulerian circuit which can be decomposed into two edge-disjoint Hamiltonian paths.

For the sake of completeness, let us exactly define the open problem.

Definition 5. *For any* $n \geq 2$, *the* de Bruijn graph *is a directed graph* $G = (V, E)$ *such that:*

- $V = \{0, \ldots, 2^n - 1\}$
- $$(i, j) \in E \iff$$
$$j \in \{(2 \cdot i) \; mod \; 2^n, \; (2 \cdot i + 1) \; mod \; 2^n\} \text{ and } j \neq i$$

1. A proof showing δ to be either 1 or 2 can be found in [17].

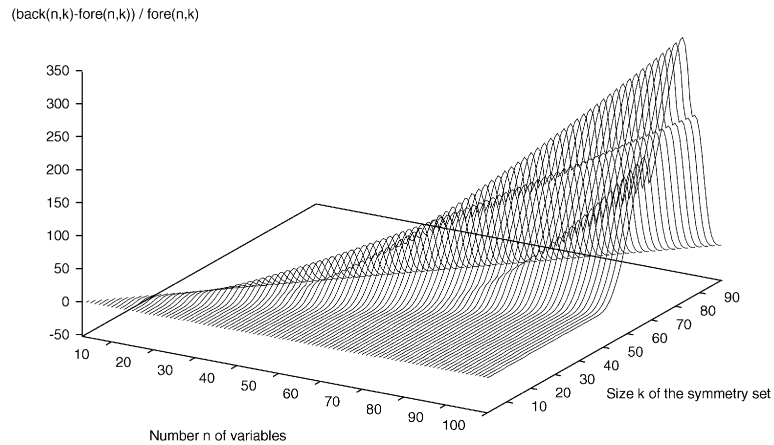


Fig. 4. Relative difference of $back(n,k)$ and $fore(n,k)$.

By [2, Theorem 6], it follows that any de Bruijn graph possesses a Eulerian circuit because it is connected and is pseudosymmetric. The open question is whether any de Bruijn graph possesses a Eulerian circuit which visits all vertices of the graph exactly once before visiting any vertex for the second time. (Note that the well-known theorem of van Aardenne-Ehrenfest and de Bruijn [36] (see also [22, p. 375]) does not solve our problem.)

7 Experimental Results

The experiments we have made are based on the upper bounds proven above.

7.1 Experiments in the Case of Exactly One Nontrivial Symmetry Set

First, we concentrate on the case of exactly one nontrivial symmetry set.

We have looked for the positions of the symmetry set in the variable order for which the upper bound is minimal. For that we have evaluated the formulas $bound(n,k,s)$ for $3 \leq n \leq 100$, $1 \leq k \leq n$, and $1 \leq s \leq n-k+1$. Fig. 2 and Fig. 3 show the areas in which fore SymOBDDs or back SymOBDDs should be used. Superimposing both figures shows that, for any n and k, either the fore or the back SymOBDD have best performance with respect to the upper bounds.

In order to better understand what happens, we have illustrated the relative difference between $back(n,k) = bound(n,k,n-k+1)$ and $fore(n,k) = bound(n,k,1)$ in Fig. 4.

TABLE 1
Percentage of Pairs (n,k) which Require Variable Order π to Obtain Minimal Upper Bounds in Case 1

Variable order π	$MinBound\_perc_\pi$
fore-fore	62%
fore-back	34%
back-back	4%

It is easy to see that if the size k of the only nontrivial symmetry set is small, i.e., $k < \frac{n}{2}$, then the difference between the size of a back SymOBDD and a fore SymOBDD is small, sometimes positive, sometimes negative. However, if $k \geq \frac{n}{2}$, the difference is always positive. This sets the trend to use fore SymOBDDs. The largest difference appears for k about $\frac{3n}{4}$. By Corollaries 3 and 4, it is easy to show that $fore\left(n, \frac{3n}{4}\right) = o\left(n 2^{\frac{n}{4}}\right)$ and $back\left(n, \frac{3n}{4}\right) = \Theta\left(n^2 2^{\frac{n}{4}}\right)$ hold.

7.2 Experiments in the Case of More than One Nontrivial Symmetry Set

The other run of experiments we made concentrates on cases where there is more than one nontrival symmetry set. We have investigated the upper bounds for the cases in which:

1. There are exactly two nontrivial symmetry sets, both of size k,
2. There are exactly two nontrivial symmetry sets λ_1 and λ_2 such that $|\lambda_2| = \frac{1}{3} |\lambda_1|$ holds, and
3. There are exactly three nontrivial symmetry sets each of size k.

In each case, a largest symmetry set is located right up at the beginning or at the end of the variable order to obtain minimal upper bound. In most cases, it is located at the

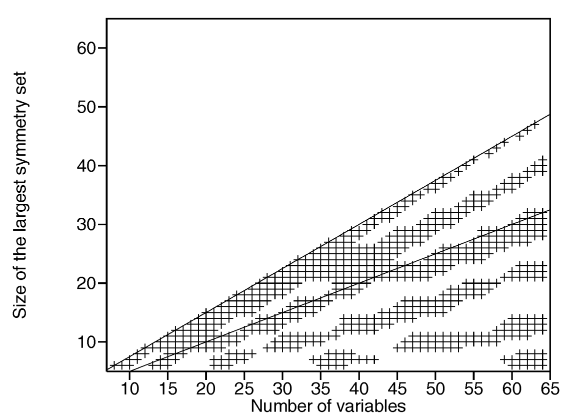


Fig. 5. Areas in which λ_1 is put right up at the beginning and λ_2 right up at the end. This area covers 50 percent of the total area.

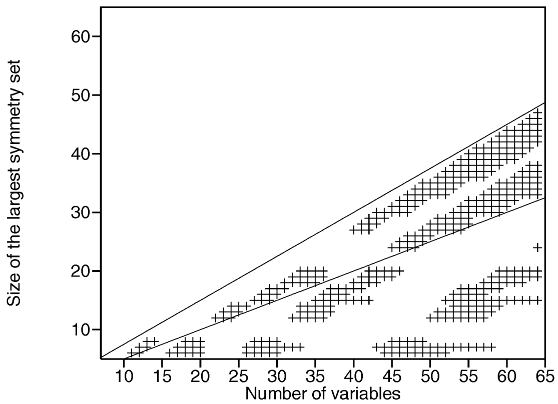


Fig. 6. Areas in which λ_1 is put right up at the beginning and λ_2 is put next to λ_1. This area covers 32 percent of the total area.

front. The other symmetry sets have to be put next to the largest symmetry set or at the other side of the variable order.

For Case 1, we evaluated the upper bounds for $4 \leq n \leq 80$ and $2 \leq k \leq 40$. The results showed that minimal upper bounds on the size of SymOBDDs can be obtained only for the following positions of the symmetry sets in the variable order:

- *fore-fore*: both symmetry sets have to be located right up at the beginning,
- *fore-back*: one symmetry set is put right up at the beginning and the other at the end,
- *back-back*: both symmetry sets are located right up at the end of the variable order.

Table 1 gives the percentage $MinBound\_perc_\pi$ of the pairs (n, k) which require the symmetry variable order $\pi \in \{fore\text{-}fore, fore\text{-}back, back\text{-}back\}$ in order to obtain minimal upper bounds. Note that, for 62 percent of the pairs (n, k) *fore-fore* SymOBDDs should be used to obtain best performance with respect to the upper bounds.

Fig. 5 and Fig. 6 illustrate the results obtained for Case 2, i.e., two symmetry sets λ_1 and λ_2 with $|\lambda_2| = \frac{1}{3}|\lambda_1|$, $4 \leq n \leq 65$, and $3 \leq |\lambda_1| \leq 48$. The percentage of pairs $(n, |\lambda_1|)$ which require that λ_1 is located right up at the beginning and λ_2 is located right up at the end of the order and the percentage of pairs where λ_1 should be located right up at the beginning and λ_2 should be put next to it, is 50 percent and 32 percent, respectively. The remaining 18 percent of the total area represents the areas covering 17 percent, where λ_1 is right up at the end and λ_2 is right up at the beginning, and the areas covering 1 percent, where λ_1 is right up at the end and λ_2 is next to it.

Note that the largest symmetry set λ_1 is located either right up at the beginning or right up at the end, in all cases.

For Case 3, where there are exactly three nontrivial symmetry sets, each of size k, upper bounds for $6 \leq n \leq 40$ and $2 \leq k \leq 13$ have been evaluated. The experiments show that:

- At 66 percent, two of the symmetry sets should be located right up at the beginning and the third one should be located right up at the end of the variable order,
- At 27 percent, all three symmetry sets should be put right up at the beginning, and
- At 7 percent, two of the symmetry sets should be located right up at the end and the third one should be put either next to them or right up at the beginning of the variable order

to obtain minimal upper bounds.

8 Conclusion

In this paper, we have proven upper bounds for the size of OBDDs of Boolean functions using symmetry properties of the functions under consideration. The experiments show that the nontrivial symmetry sets should be located right up at the beginning or the end of the variable orders to obtain best performance with respect to the upper bounds. It seems to be that, at the beginning and at the end of the variable order, respectively, the symmetry sets should be ordered by their sizes. Of course, these properties may not hold for specific Boolean functions. However, the investigations made in this paper set trends as to where to place the symmetry sets in variable orders in order to obtain efficient symmetry ordered OBDDs.

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